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# A Class of Non-Symmetric Semi-Classical Linear Forms

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**Abstract:** We show that if $v$ is a regular semi-classical form (linear functional), then the form $u$ defined by $(x-\tau^2)\sigma u = -\lambda v$ and $\sigma(x-\tau)u = 0$ where $\sigma u$ is the even part of $u$, is also regular and semi-classical form for every complex $\lambda$ except for a discrete set of numbers depending on $v$. We give explicitly the recurrence coefficients and the structure relation coefficients of the orthogonal polynomials sequence associated with $u$ and the class of the form $u$ knowing that of $v$. We conclude with illustrative examples.

**Key words:** Orthogonal polynomials; Semi-classical linear forms; Integral representation; Structure relation

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## 1. INTRODUCTION

Semi-classical orthogonal polynomials (O.P) were introduced in [14]. They are naturel generalization of the classical polynomials (Hermite, Laguerre, Jacobi and Bessel). Maroni [8, 10] has worked on the linear form of moments and has given a unified theory of this kind of polynomials. The form $u$ is called semi-classical form if its formal Stieltjes function $S(u)(z)$ satisfies the Riccati differential equation $\Phi(z)S'(u)(z) = C_0(z)S(u)(z) + D_0(z)$, where $\Phi \neq 0, C_0$ and $D_0$ are polynomials.

In [5, 7], the authors determine all the semi-classical monic orthogonal polynomials sequence (MOPS) of class one satisfying a three terms recurrence relation with $\beta_n = (-1)^n\tau$, $n \geq 0$, $\tau \in \mathbb{C} - \{0\}$. See also [1] for a special case.

The whole idea of the following work is to build a new construction process of semi-classical form, which has not yet been treated in the literature on semi-classical polynomials. The problem we tackle is as follows.

We study the form $u$, fulfilling

$$(x-\tau^2)\sigma u = -\lambda v, \quad \lambda \neq 0, \quad (u)_{2n+1} = \tau (u)_{2n},$$

where $\sigma u$ is the even part of $u$, $\tau \in \mathbb{C}$ and $v$ is a given semi-classical form.

This paper is arranged in sections: The first provides a focus on the preliminary results and notations used in the sequel. We will also give the regularity condition and the coefficients of the three-term recurrence relation satisfied by the new family of O.P. In the second, we compute the exact class of the semi-classical form obtained by the above modification and the structure relation of the O.P. sequence relatively to the form $u$ will follow. In the final section, we apply our results to some examples. The regular linear functional found in the examples are semi-classical linear functional of class $\tilde{s} \in \{1,2\}$ and we present their integral representations.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$, the moments of $v$. For any form $v$ and any polynomial $h$ let $Dv = v'$, $hv$, $\delta_c$, and $(x-c)^{-1}v$ be the forms defined by:

$$\langle v', f\rangle := -\langle v, f'\rangle, \quad \langle hv, f\rangle := \langle v, hf\rangle, \quad \langle \delta_c, f\rangle := f(c),$$

and

$$\big\langle (x-c)^{-1}v, f\big\rangle := \langle v, \theta_c f\rangle,$$

where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$(x-c)^{-1}((x-c)v) = v - (v)_0\delta_c, \tag{1}$$

$$(x-c)((x-c)^{-1}u) = v. \tag{2}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$. Then, we define the even part $\sigma v$ of $v$ by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$.
Therefore, we have [6, 9]

$$f(x)(\sigma v) = \sigma(f(x^2)v), \tag{3}$$

$$(\sigma v)_n = (v)_{2n}, \quad n \geq 0. \tag{4}$$

The form $v$ will be called regular if we can associate with it a polynomial sequence $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\ \big)$ such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m}, \quad n,m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

Then $\deg(S_n) = n,\ \ n \geq 0$, and we can always suppose each $S_n$ monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to $v$. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation $\big($see, for instance, the monograph by Chihara [6]$\big)$

$$\begin{aligned} S_{n+2}(x) &= (x-\xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x), \quad n \geq 0, \\ S_1(x) &= x - \xi_0, \qquad S_0(x) = 1, \end{aligned} \tag{5}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C} - \{0\}, \quad n \geq 0$, by convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation [6]

$$\begin{aligned} &S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x), \quad n \geq 0, \\ &S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1, \quad \big(S_{-1}^{(1)}(x) = 0\big). \end{aligned} \tag{6}$$

Another important representation of $S_n^{(1)}$ is, (see [6])

$$S_n^{(1)}(x) := \Big\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta} \Big\rangle, \, n \geq 0. \tag{7}$$

Also, let $\{S_n(., \mu)\}_{n\geq 0}$ be co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [6]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)}, \quad n \geq 0. \tag{8}$$

We recall that a form $v$ is called symmetric if $(v)_{2n+1} = 0$, $n \geq 0$. The conditions $(v)_{2n+1} = 0$, $n \geq 0$ are equivalent to the fact the corresponding MOPS $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (5) with $\xi_n = 0$, $n \geq 0$ [6].

Now let $v$ be a regular, normalized form (i.e. $(v)_0 = 1$) and $\{S_n\}_{n\geq 0}$ be its corresponding sequence of polynomials. For a $\tau \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, we can define a new form $u$ as following:

$$(u)_{2n+2} - \tau^2(u)_{2n} = -\lambda(v)_n, \ (u)_{2n+1} = \tau(u)_{2n}, \ (u)_0 = 1, \quad n \geq 0. \tag{9}$$

Equivalently,

$$(x - \tau^2)\sigma u = -\lambda v, \qquad \sigma\big((x - \tau)u\big) = 0. \tag{10}$$

From (1) and (10), we have

$$\sigma u = -\lambda(x - \tau^2)^{-1}v + \delta_{\tau^2}. \tag{11}$$

*Remarks.*

i) (10) is equivalent to

$$(x^2 - \tau^2)u = -\lambda w, \tag{12}$$

where the form $w$ defined by

$$\sigma w = v, \quad \sigma(x - \tau)w = 0.$$

Notice that $w$ is not necessarily a regular form in the problem understudy. In [2], the authors have solved where $w$ is regular and $\tau = 0$ and in [3], the problem (12) is solved when $\tau \neq 0$ and $w$ is regular.

ii) The case $\tau = 0$ is treated in [13], so henceforth we assume $\tau \neq 0$.

**Proposition 1.** *The form $u$ is regular if and only if $\lambda \neq \lambda_n, n \geq 0$ where*

$$\lambda_0 = 0, \quad \lambda_{n+1} = \frac{S_{n+1}(\tau^2)}{S_n^{(1)}(\tau^2)}, n \geq 0. \tag{13}$$

To prove the above proposition, we need the following lemma:

**Lemma 2.** [9] *The form $u$ defined by* (10) *is regular if and only if $\sigma u$ and $(x-\tau^2)\sigma u$ are regular.*

**Proof of Proposition 1.** We have $u$ is defined by (10). Then, according to Lemma 2, $u$ is regular if and only if $(x-\tau^2)\sigma u$ and $\sigma u$ are regular. But $(x-\tau^2)\sigma u = -\lambda v$ is regular since $\lambda \neq 0$ and $v$ is regular. So $u$ is regular if and only if $\sigma u = -\lambda(x-\tau^2)^{-1}\sigma v + \delta_{\tau^2}$ is regular. Or, $\{S_n\}_{n\geq 0}$ is the corresponding orthogonal sequence to $v$, and it was shown in [11] that $\sigma u = -\lambda(x-\tau^2)^{-1}v + \delta_{\tau^2}$ is regular if and only if $\lambda \neq 0$, and $S_n(\tau^2, \lambda) \neq 0, n \geq 0$. Then we deduce the desired result.

When $u$ is regular let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} Z_{n+2}(x) &= (x-(-1)^{n+1}\tau)Z_{n+1}(x) - \gamma_{n+1}Z_n(x), \quad n \geq 0, \\ Z_1(x) &= x-\tau, \quad Z_0(x) = 1. \end{aligned} \tag{14}$$

Let us consider its quadratic decomposition [6, 9]:

$$Z_{2n}(x) = P_n(x^2), \quad Z_{2n+1}(x) = (x-\tau)R_n(x^2). \tag{15}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to $\sigma u$ and $(x-\tau^2)\sigma u$.

From (10), we have

$$R_n(x) = S_n(x), \quad n \geq 0. \tag{16}$$

**Proposition 3.** *We may write*

$$\gamma_1 = -\lambda, \quad \gamma_{2n+2} = a_n, \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n}, \quad n \geq 0, \tag{17}$$

*where*

$$a_n = -\frac{S_{n+1}(\tau^2, \lambda)}{S_n(\tau^2, \lambda)}, \quad n \geq 0. \tag{18}$$

For the proof, we need the following lemma:

**Lemma 4.** [4] *We have*

$$Z^{(1)}_{2n}(x) = R_n(x^2, \lambda), \quad Z^{(1)}_{2n+1}(x) = (x+\tau)P^{(1)}_n(x^2), \quad n \geq 0. \tag{19}$$

**Proof of Proposition 3.** Using (10) and the condition $\langle u, Z_2\rangle = 0$, we obtain $\gamma_1 = -\lambda$.

From (6) and (14) where $n \longrightarrow 2n$ and taking (16) and (19) into account, we get

$$S_{n+1}\left(x^2, -\gamma_1\right) = (x-\tau)Z^{(1)}_{2n+1}(x) - \gamma_{2n+2}S_n\left(x^2, -\gamma_1\right).$$

Substituting $x$ by $\tau$ in the above equation, we obtain $\gamma_{2n+2} = a_n$.

From (14), we have

$$\gamma_{2n+2}\gamma_{2n+3} = \frac{\left\langle u, Z^2_{2n+2}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle}\frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+2}\right\rangle} = \frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle}. \tag{20}$$

Using (5), (10) and (15) – (16), Equation (20) becomes

$$\gamma_{2n+2}\gamma_{2n+3} = \rho_{n+1}, \tag{21}$$

then we deduce $\gamma_{2n+3} = \dfrac{\rho_{n+1}}{a_n}$.

We suppose that the form $v$ has the following integral representation:

$$\langle v, f\rangle = \int_0^{+\infty} V(x)f(x)\mathrm{d}x,\ f \in \mathcal{P}, \text{with } (v)_0 = \langle v, f\rangle = \int_0^{+\infty} V(x)\mathrm{d}x,$$

where $V$ is a locally integrable function with rapid decay and continuous at the point $x = \tau^2$.

It is obvious that

$$f(x) = f^e(x^2) + xf^o(x^2),\ f \in \mathcal{P}.$$

Therefore,

$$\langle u, f(x)\rangle = \langle u, f^e(x^2) + \tau f^o(x^2)\rangle = \langle \sigma u, f^e(x) + \tau f^o(x)\rangle,$$

since $u$ satisfies (10).

Using (11) and taking into account that $f^e(\tau^2) + \tau f^o(\tau^2) = f(\tau)$, we obtain

$$\begin{aligned}\langle u, f\rangle = f(\tau)\Big\{1{+}\lambda P\int_{-\infty}^{+\infty} \frac{V(x)}{x-\tau^2}\chi_{[0,+\infty[}(x)\mathrm{d}x\Big\}\\ -\lambda P\int_{-\infty}^{+\infty} \frac{V(x)}{x-\tau^2}\chi_{[0,+\infty[}(x)(f^e+\tau f^o)(x)\mathrm{d}x,\end{aligned} \tag{22}$$

where

$$P\int_{-\infty}^{+\infty} \frac{V(x)}{x-\tau^2}f(x) = \lim_{\epsilon\to 0}\Big\{\int_{-\infty}^{\tau^2-\varepsilon} \frac{V(x)}{x-\tau^2}f(x)\mathrm{d}x + \int_{\tau^2+\epsilon}^{+\infty} \frac{V(x)}{x-\tau^2}f(x)\mathrm{d}x\Big\}$$

and $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$, i.e. $\chi_{[a,b]}(x) = 1$ when $x \in [a, b]$ and zero otherwise.

Using the fact that $f^e(x) = \frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$ and $f^o(x) = \frac{f(\sqrt{x})-f(-\sqrt{x})}{2\sqrt{x}}$ for $x > 0$ and making the change of variables $t = \sqrt{x}$, we get

$$\begin{aligned}P\int_{-\infty}^{+\infty} \frac{V(x)}{x-\tau^2}\chi_{[0,+\infty[}(x)(f^e+\tau f^o)(x)\mathrm{d}x =&P\int_{-\infty}^{+\infty} \frac{V(t^2)}{t-\tau}\chi_{[0,+\infty[}(t)f(t)\mathrm{d}t\\ &+P\int_{-\infty}^{+\infty} \frac{V(t^2)}{t+\tau}\chi_{[0,+\infty[}(t)f(-t)\mathrm{d}t.\end{aligned}$$

Inserting the last equation into (22), we get after a change variables in the obtained equation

$$\begin{aligned}\langle u, f\rangle = f(\tau)\Big\{1{+}\lambda P\int_{-\infty}^{+\infty} \frac{V(t)}{t-\tau^2}\chi_{[0,+\infty[}(t)\mathrm{d}t\Big\}\\ +\lambda P\int_{-\infty}^{+\infty} \frac{V(t^2)}{t-\tau}\chi_{]-\infty,0]}(t)f(t)\mathrm{d}t\\ -\lambda P\int_{-\infty}^{+\infty} \frac{V(t^2)}{t-\tau}\chi_{[0,+\infty[}(t)f(t)\mathrm{d}t.\end{aligned} \tag{23}$$

## 2. THE SEMI-CLASSICAL CASE

Let us recall that a form $v$ is called semi-classical when it is regular and satisfies a linear non-homogeneous differential equation [10]

$$\Phi(z)S'\big(v\big)(z) = C_0(z)S\big(v\big)(z) + D_0(z), \tag{24}$$

where $\Phi$ monic, $C_0$ and $D_0$ are polynomials with

$$S(v)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}}, \tag{25}$$

and

$$D_0(x) = -\big(v\theta_0\Phi\big)'(x) + \big(v\theta_0 C_0\big)(x). \tag{26}$$

It was shown in [10] that equation (24) is equivalent to

$$\big(\Phi(x)v\big)' + \Psi(x)v = 0 \tag{27}$$

with

$$\Psi(x) = -\Phi'(x) - C_0(x). \tag{28}$$

The triple $(\Phi, C_0, D_0)$ of the differential equation is not unique, then (24) can simplified if and only if there exists a root $c$ of $\Phi$ such that $C_0(c) = 0$ and $D_0(c) = 0$. Then $v$ fulfils the differential equation

$$(\theta_c\Phi)(z)S'\big(v\big)(z) = (\theta_c C_0)(z)S\big(v\big)(z) + (\theta_c D_0)(z).$$

We call the class of the linear form $v$, the minimum value of the integer $\max\big(\deg(\Phi) - 2, \deg(C_0) - 1\big)$ for all triples satisfying (24).
The class of the semi-classical form $v$ is $s = \max\big(\deg(\Phi) - 2, \deg(C_0) - 1\big)$ if and only if the following condition is satisfied [8]

$$\prod_{c\in\mathcal{Z}} (|C_0(c)| + |D_0(c)|) \neq 0, \tag{29}$$

where $\mathcal{Z}$ denotes the set of zeros of $\Phi$.

The corresponding orthogonal sequence $\{S_n\}_{n\geq 0}$ is also called semi-classical of class $s$.

The semi-classical character is invariant by shifting. Indeed, the shifted form $\hat{v} = (h_{a^{-1}} o t_{-b})v,\ a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}$ satisfies

$$\hat{\Phi}(z)S'\big(\hat{v}\big)(z) = \hat{C}_0(z)S\big(\hat{v}\big)(z) + \hat{D}_0(z), \tag{30}$$

with

$$\hat{\Phi}(z) = a^{-k}\Phi(az + b), \qquad \hat{C}_0(z) = a^{1-k}C_0(az + b),$$
$$\hat{D}_0(z) = a^{2-k}D_0(az + b), \qquad k = \deg(\Phi).$$

The forms $t_b v$ (translation of $v$) and $h_a v$ (dilation of $v$) are defined by

$$\langle t_b v, f\rangle := \langle v, f(x + b)\rangle, \quad \langle h_a v, f\rangle := \langle v, f(ax)\rangle, \quad f \in \mathcal{P}.$$

The sequence $\{\hat{S}_n(x) = a^{-n}S_n(ax+b)\}_{n\geq 0}$ is orthogonal with respect to $\hat{v}$ and fulfils (5) with

$$\hat{\xi}_n = \frac{\xi_n - b}{a}, \quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2}, \quad n \geq 0. \tag{31}$$

In the sequel the form $v$ will be supposed semi-classical linear form of class $s$ satisfying (24) and (27) and using a dilation in the variable $\tau$, we can take him equal to one.

**Proposition 5.** *For every $\lambda \in \mathbb{C} - \{0\}$ such that $S_n(1,\lambda) \neq 0, n \geq 0$, the form $u$ defined by* (10) *is regular and semi-classical. It satisfies*

$$\tilde{\Phi}(z)S'\big(u\big)(z) = \tilde{C}_0(z)S\big(u\big)(z) + \tilde{D}_0(z), \tag{32}$$

*where*

$$\begin{cases} \tilde{\Phi}(z) = (z-1)\Phi(z^2), \\ \tilde{C}_0(z) = 2z(z-1)C_0(z^2) - \Phi(z^2), \\ \tilde{D}_0(z) = -2z\big(\lambda D_0(z^2) - C_0(z^2)\big), \end{cases} \tag{33}$$

and $u$ is of class $\tilde{s}$ such that $\tilde{s} \leq 2s+3$.

*Proof.* From (10) and (25), we have

$$S\big(v\big)(z^2) = -\lambda^{-1}(z-1)S\big(u\big)(z) - \lambda^{-1}. \tag{34}$$

Make a change of variable $z \longrightarrow z^2$ in (24), multiply by $-2\lambda z$ and substitute (34) in the obtained equation, we get (32) – (33).

Then, $\deg(\tilde{\Phi}) \leq 2s+5$ and $\deg(\tilde{C}_0) \leq 2s+4$.

Thus, $\tilde{s} = \max(\deg(\tilde{\Phi})-2, \deg(\tilde{C}_0)-1) \leq 2s+3$.

As an immediate consequence of (32) – (33), the form $u$ satisfies the functional equation

$$\big(\tilde{\Phi}u\big)' + \tilde{\Psi}u = 0, \tag{35}$$

where $\tilde{\Phi}$ is the polynomial defined by (33) and

$$\tilde{\Psi}(x) = -\tilde{\Phi}'(x) - \tilde{C}_0(x) = 2x(x-1)\Psi(x^2). \tag{36}$$

**Proposition 6.** *The class of $u$ depends only on the zeros $x=0$ and $x=1$ of $\tilde{\Phi}$.*

*Proof.* Since $v$ is a semi-classical form of class $s$, $S(v)(z)$ satisfies (24), where the polynomials $\Phi$, $C_0$ and $D_0$ are coprime. Let $\tilde{\Phi}$, $\tilde{C}_0$ and $\tilde{D}_0$ be as in Proposition 5. Let $d$ be a zero of $\tilde{\Phi}$ different from 0 and 1, this implies that $\Phi(d^2) = 0$. We know that $|C_0(d^2)| + |D_0(d^2)| \neq 0$

i) if $C_0(d^2) \neq 0$, then $\tilde{C}_0(d) \neq 0$,

ii) if $C_0(d^2) = 0$, then $\tilde{D}_0(d) \neq 0$, whence $|\tilde{C}_0(d)| + |\tilde{D}_0(d)| \neq 0$.

Concerning the class of $u$, we have the following result (see Proposition 8). But first, let us this technical lemma.

**Lemma 7.** *Let $X(z) = C_0(z) - \lambda D_0(z)$ and $Y(z) = C_0(z) - \Phi^{'}(z)$, where the polynomials $\Phi$, $C_0$ and $D_0$ are defined in (24). We have the following properties.*

$R_1$. *The equation $(32) - (33)$ is irreducible in 0 if and only if $\Phi(0) \neq 0$.*

$R_2$. *The equation $(32) - (33)$ is divisible by $z$ but not by $z^2$ if and only if $\Phi(0) = 0$.*

$R_3$. *The equation $(32) - (33)$ is irreducible in 1 if and only if*

$$(\Phi(1), X(1)) \neq (0,0).$$

$R_4$. *The equation $(32) - (33)$ is divisible by $z-1$ and not by $(z-1)^2$ if and only if*

$$(\Phi(1), X(1)) = (0,0) \text{ and } (X^{'}(1), Y(1)) \neq (0,0).$$

$R_5$. *The equation $(32) - (33)$ is divisible by $(z-1)^2$ and not by $(z-1)^3$ if and only if*

$$(\Phi(1), X(1)) = (X^{'}(1), Y(1)) = (0,0).$$

*Proof.* From (33), we have $\tilde{\Phi}(0) = -\Phi(0)$. So by virtue of (29), we get $R_1$.

Now, if $\Phi(0) = 0$, the equation $(32) - (33)$ is divisible by $z$ according to (29). Thus $S(u)(z)$ satisfies (32) with

$$\begin{cases} \tilde{\Phi}(z) = z(z-1)(\theta_0\Phi)(z^2), \\ \tilde{C}_0(z) = 2(z-1)C_0(z^2) - z(\theta_0\Phi)(z^2), \\ \tilde{D}_0(z) = 2C_0(z^2) - 2\lambda D_0(z^2). \end{cases} \tag{37}$$

Then, $\tilde{C}_0(0) = -C_0(0)$. If $C_0(0) = 0$, thus the equation $(32) - (37)$ is irreducible in 0. If $C_0(0) = 0$, so from (37), we obtain $\tilde{D}_0(0) = -2\lambda D_0(0) \neq 0$ since $v$ is semi-classical form of class $s$ and so satisfies (29). Therefore, we deduce $R_2$.

From (33), we get $\tilde{C}_0(1) = -\Phi(1)$ and $\tilde{D}_0(1) = 2X(1)$.
We can deduce that $|\tilde{C}_0(1)| + |\tilde{D}_0(1)| \neq 0$ if and only if $(\Phi(1), X(1)) \neq (0,0)$. Thus $R_3$ is proved.

If $(\Phi(1), X(1)) = (0,0)$, then the equation $(32) - (33)$ can be divided by $z-1$ according to (29). In this case, $S(u)(z)$ satisfies (32) with

$$\begin{cases} \tilde{\Phi}(z) = \Phi(z^2), \\ \tilde{C}_0(z) = 2zC_0(z^2) - (z+1)(\theta_1\Phi)(z^2), \\ \tilde{D}_0(z) = 2C_0(z^2) - 2\lambda D_0(z^2) + 2(z+1)\big(\theta_1(C_0 - \lambda D_0)\big)(z^2). \end{cases} \tag{38}$$

Substituting $z$ by 1 in (38), we obtain $\tilde{C}_0(1) = Y(1)$ and $\tilde{D}_0(1) = X^{'}(1)$. Then $(32) - (38)$ is irreducible in 1 if and only if $(X^{'}(1), Y(1)) \neq (0,0)$. Hence $R_4$.

If $(X^{'}(1), Y(1)) \neq (0,0)$, then the equation $(32) - (38)$ can be divided by $z-1$ according to (29). Therefore $S(u)(z)$ satisfies (32) with

$$\begin{cases} \tilde{\Phi}(z) = (z+1)(\theta_1\Phi)(z^2), \\ \tilde{C}_0(z) = 2C_0(z^2) + 2(z+1)\big(\theta_1\big(C_0 - \theta_1\Phi\big)\big)(z^2) - (\theta_1\Phi)(z^2), \\ \tilde{D}_0(z) = -2(z+2)\big(\theta_1(\lambda D_0 - C_0)\big)(z^2) - 4\big(\theta_1^2(\lambda D_0 - C_0)\big)(z^2). \end{cases} \tag{39}$$

From the above equation, we have $\tilde{\Phi}(1) = 2\Phi^{'}(1) = 0$. If $\Phi^{'}(1) = 0$, then from the condition $Y(1) = 0$ we obtain $C_0(1) = 0$. Thus from the last result and the condition $X(1) = 0$, we get $D_0(1) = 0$. Impossible, since $v$ is semi-classical form of class $s$ and so satisfies (29). Thus $R_5$ is proved.

**Proposition 8.** *Under the conditions of Proposition 5, for the class of $u$, we have the two different cases:*

*1)* $\Phi(0) \neq 0$.

*i)* $\tilde{s} = 2s + 3$ *if* $(\Phi(1), X(1)) \neq (0,0)$.
*ii)* $\tilde{s} = 2s + 2$ *if* $(\Phi(1), X(1)) = (0,0)$ *and* $(Y(1), X^{'}(1)) \neq (0,0)$.
*iii)* $\tilde{s} = 2s + 1$ *if* $(\Phi(1), X(1)) = (Y(1), X^{'}(1)) = (0,0)$.

*2)* $\Phi(0) = 0$.

*i)* $\tilde{s} = 2s + 2$ *if* $(\Phi(1), X(1)) \neq (0,0)$.
*ii)* $\tilde{s} = 2s + 1$ *if* $(\Phi(1), X(1)) = (0,0)$ *and* $(Y(1), X^{'}(1)) \neq (0,0)$.
*iii)* $\tilde{s} = 2s$ *if* $(\Phi(1), X(1)) = (Y(1), X^{'}(1)) = (0,0)$.

*Proof.* From Proposition 6, the class of $u$ depends only on the zeros 0 and 1. For the zero 0 we consider the following situations:

**A)** $\Phi(0) \neq 0$. In this case the equation (32) – (33) is irreducible in 0 according to $R_1$. But what about the zero 1?

We will analyze the following cases:

i) $(\Phi(1), X(1)) \neq (0,0)$, the equation (32) – (33) is irreducible in 1 according to $R_3$. Then (32) – (33) is irreducible and $\tilde{s} = 2s + 3$. Thus we proved 1) i).
ii) $(\Phi(1), X(1)) = (0,0)$ and $(Y(1), X^{'}(1)) \neq (0,0)$.

From $R_4$., (32) – (33) is divisible by $z - 1$ but not by $(z-1)^2$ and thus the order of the class of $u$ decreases in one unit. In fact, $S(u)(z)$ satisfies the irreducible equation (32) – (38) and then $\tilde{s} = 2s + 2$ and 1) ii) is also proved.

iii) $(\Phi(1), X(1)) = (Y(1), X^{'}(1)) = (0,0)$.

From $R_5$., (32) – (33) is divisible by $(z-1)^2$ but not by $(z-1)^3$ and thus the order of the class of $u$ decreases in two units. In fact, $S(u)(z)$ satisfies the irreducible equation (32) – (39) and then $\tilde{s} = 2s + 1$. Thus 1) iii) is proved.

**B)** $\Phi(0) \neq 0$. In this condition, (32) – (33) is divisible by $z$ but not by $z^2$ according to $R_2$. But what about the zero 1?

We have the three following cases:

i) $(\Phi(1), X(1)) \neq (0,0)$, the equation (32) – (33) is irreducible in 1 according to $R_3$. Then $S(u)(z)$ satisfies the irreducible equation (32) – (37) and then $\tilde{s} = 2s + 2$. Thus we proved 2) i).
ii) $(\Phi(1), X(1)) = (0,0)$ and $(Y(1), X^{'}(1)) \neq (0,0)$.

From $R_4$., (32) – (33) is divisible by $z - 1$ but not by $(z-1)^2$ and thus the order of the class of $u$ decreases in one unit. In fact, $S(u)(z)$ satisfies the irreducible Equation (32) with

$$\begin{cases} \tilde{\Phi}(z) = z(\theta_0\Phi)(z^2), \\ \tilde{C}_0(z) = 2C_0(z^2) - (\theta_0\Phi)(z^2) - (z+1)(\theta_0\theta_1\Phi)(z^2), \\ \tilde{D}_0(z) = -2(z+1)\big(\theta_1(\lambda D_0 - C_0)\big)(z^2). \end{cases} \tag{40}$$

Thus $\tilde{s} = 2s + 1$ and 2) ii) is proved.

iii) $(\Phi(1), X(1)) = (Y(1), X^{'}(1)) = (0, 0)$.

From $R_5$., (32) – (33) is divisible by $(z-1)^2$ but not by $(z-1)^3$. So, $S(u)(z)$ satisfies the irreducible Equation (32) with

$$\begin{cases} \tilde{\Phi}(z) = (z+1)(\theta_0\theta_1\Phi)(z^2) + (\theta_0\Phi)(z^2), \\ \tilde{C}_0(z) = 2(z+1)\big(\theta_1\big(C_0 - \theta_0\theta_1\Phi\big)\big)(z^2) - (z+2)(\theta_0\theta_1\Phi)(z^2), \\ \tilde{D}_0(z) = -2\big(\theta_1(\lambda D_0 - C_0)\big)(z^2) - 4(z+1)\big(\theta_1^2(\lambda D_0 - C_0)\big)(z^2). \end{cases} \tag{41}$$

Therefore $\tilde{s} = 2s$ and 2) iii) is also proved.

Note that the sequence of orthogonal polynomials (OPS) relatively to a semi-classical form has a structure relation (written in a compact form)[10]. Then, if we consider that the form $v$ is semi-classical, its OPS $\{S_n\}_{n\geq 0}$ fulfils the following structure relation:

$$\Phi(x)S'_{n+1}(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)S_{n+1}(x) - \rho_{n+1}D_{n+1}(x)S_n(x), \quad n \geq 0, \tag{42}$$

with

$$\begin{cases} C_{n+1}(x) = -C_n(x) + 2(x-\xi_n)D_n(x), \quad n \geq 0, \\ \rho_{n+1}D_{n+1}(x) = -\Phi(x) + \rho_n D_{n-1}(x) - (x-\xi_n)C_n(x) \\ \qquad\qquad + (x-\xi_n)^2 D_n(x), \quad n \geq 0, \end{cases} \tag{43}$$

where $\Phi$, $C_0(x)$ and $D_0(x)$ are the same polynomials as in (24); $\xi_n, \rho_n$ are the coefficients of the three term recurrence relation (5). Notice that $D_{-1}(x) = 0, \deg C_n \leq s+1$ and $\deg D_n \leq s, n \geq 0$ [10].

According to Proposition 5, the form $u$ is also semi-classical and its OPS $\{Z_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{Z_n\}_{n\geq 0}$ fulfils

$$\tilde{\Phi}(x)Z'_{n+1}(x) = \frac{1}{2}\big(\tilde{C}_{n+1}(x) - \tilde{C}_0(x)\big)Z_{n+1}(x) - \gamma_{n+1}\tilde{D}_{n+1}(x)Z_n(x), \quad n \geq 0, \tag{44}$$

with

$$\begin{cases} \tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2(x-(-1)^n)\tilde{D}_n(x), \quad n \geq 0, \\ \gamma_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \gamma_n\tilde{D}_{n-1}(x) - (x-(-1)^n)\tilde{C}_n(x) \\ \qquad\qquad + (x-(-1)^n)^2\tilde{D}_n(x), \quad n \geq 0, \end{cases} \tag{45}$$

where $\tilde{\Phi}$, $\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are the same polynomials as in Equation (32).

We are going to establish the expression of $\tilde{C}_n$ and $\tilde{D}_n$ , $n \geq 0$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

**Proposition 9.** *The sequence* $\{Z_n\}_{n\geq 0}$ *fulfils* (44) *with (for* $n \geq 0$*)*

$$\begin{cases} \tilde{C}_{2n+1}(x) = \Phi(x^2) + 2x(x-1)C_n(x^2) + 4\gamma_{2n+1}x(x-1)D_n(x^2), \\ \tilde{D}_{2n+1}(x) = 2x(x-1)^2 D_n(x^2). \end{cases} \tag{46}$$

$$\begin{cases} \tilde{C}_{2n+2}(x) = -\Phi(x^2) + 2x(x-1)C_{n+1}(x^2) + 4x(x-1)\gamma_{2n+2}D_n(x^2), \\ \tilde{D}_{2n+2}(x) = 2x\gamma_{2n+2}D_n(x^2) + 2x\gamma_{2n+3}D_{n+1}(x^2) + 2xC_{n+1}(x^2). \end{cases} \tag{47}$$

$\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are given by (33) and $\gamma_{n+1}$ by (17).

*Proof.* Change $x \longrightarrow x^2$, $n \longrightarrow n-1$ in (42) and multiply by $2x(x-1)^2$ we obtain by taking (15) – (16) into account,

$$(x-1)\Phi(x)Z'_{2n+3}(x) = \left\{x(x-1)\Big(C_{n+1}(x^2) - C_0(x^2)\Big) + \Phi(x^2)\right\}Z_{2n+3}(x) \\ - 2\rho_{n+1}x(x-1)D_{n+1}(x^2)Z_{2n+1}(x), \quad n \geq 0.$$

Using (14) and (21) where $n \longrightarrow 2n$, the last equation becomes

$$\tilde{\Phi}(x)Z'_{2n+3}(x) = \Bigg\{x(x-1)\Big(C_{n+1}(x^2) - C_0(x^2) + 2x(x-1)\gamma_{2n+3}D_{n+1}(x^2)\Big) \\ + \Phi(x^2)\Bigg\}Z_{2n+3}(x) - 2\gamma_{2n+3}x(x-1)^2D_{n+1}(x^2)Z_{2n+2}(x),\ n \geq 0.$$

From (44) and the above equation, we have for $n \geq 0$

$$\left\{\frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2} - X_{n+1}(x)\right\} Z_{2n+3}(x) = \gamma_{2n+3}\left\{\tilde{D}_{2n+3} - Y_{n+1}(x)\right\} Z_{2n+2}(x),$$

with for $n \geq 0$

$$\begin{cases} X_n(x) = \big(C_n(x^2) - C_0(x^2) + 2\gamma_{2n+1}D_n(x^2)\big)\, x(x-1) + \Phi(x^2), \\ Y_n(x) = 2x(x-1)^2D_n(x^2). \end{cases}$$

$Z_{2n+3}$ and $Z_{2n+2}$ have no common zeros, then $Z_{2n+3}$ divides $Y_{n+1}(x) - \tilde{D}_{2n+3}(x)$, which is a polynomial of degree at most equal to $2s+3$. Then we have necessarily $Y_{n+1}(x) - \tilde{D}_{2n+3}(x) = 0$ for $n > s$, and also

$$X_n(x) = \frac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2}, \quad n > s.$$

Therefore,

$$\tilde{C}_{2n+3}(x) = 2X_{n+1}(x) + \tilde{C}_0(x) \quad \text{and} \quad \tilde{D}_{2n+3} = Y_{n+1}(x), \quad n > s.$$

Then, by (33), we get (46) for $n > s$.

By virtue of the recurrence relation (43) and (33), we can easily prove by induction that the system (46) is valid for $0 \leq n \leq s$. Hence (46) is valid for $n \geq 0$.

After a derivation of (14) where $n \to 2n+1$ multiplying by $(x-1)\Phi(x^2)$ and using (44), we obtain

$$(x-1)^2\Phi(x^2)Z'_{2n+2}(x) = \frac{\tilde{C}_{2n+3}(x) - \tilde{C}_0(x)}{2}Z_{2n+3}(x) \\ - \gamma_{2n+3}\tilde{D}_{2n+3}(x)Z_{2n+2}(x) - (x-1)\Phi(x^2)Z_{2n+2}(x) \\ + \gamma_{2n+2}\Big\{\frac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2}Z_{2n+1}(x) - \gamma_{2n+1}\tilde{D}_{2n+1}(x)Z_{2n}(x)\Big\}.$$

Applying the recurrence relation (14), we get

$$\begin{aligned}(x-1)^2\Phi(x^2)Z'_{2n+2}(x) = \Big\{(x-1)\frac{\tilde{C}_{2n+3}(x)-\tilde{C}_0(x)}{2} + \gamma_{2n+2}\tilde{D}_{2n+1}(x)\\ - \gamma_{2n+3}\tilde{D}_{2n+3}(x) - (x-1)\Phi(x^2)\Big\}Z_{2n+2}(x)\\ - \gamma_{2n+2}\Big\{\frac{\tilde{C}_{2n+3}(x)-\tilde{C}_{2n+1}(x)}{2} + (x+1)\tilde{D}_{2n+1}(x)\Big\}Z_{2n+1}(x).\end{aligned}$$

Now, using (44) and taking into account the fact that $Z_{2n+2}(x)$ and $Z_{2n+1}(x)$ are coprime, we get from the last equation after simplification by $x-1$ (47) for $n > s$. Finally, by virtu of the recurrence relation (43) and (46) with $n = 0$, we can easily prove by induction that the system (47) is valid for $0 \le n \le s$.

## 3. ILLUSTRATIVE EXAMPLES

(1) We study the problem (10), with $v := \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the Laguerre form. This form has the following integral representation [10]

$$\langle v, f\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty} x^\alpha e^{-x} f(x)\mathrm{d}x,\ \Re(\alpha) > -1,\ f\in\mathcal{P}. \tag{48}$$

Thus, using (23), we obtain the following integral representation of $u$

$$\begin{aligned}\langle u, f\rangle = f(1)\Big\{1 + \lambda P\int_{-\infty}^{+\infty}\frac{x^\alpha e^{-x}}{x-1}\chi_{[0,+\infty[}(x)\mathrm{d}x\Big\}\\ + \lambda\int_{-\infty}^{0}\frac{x^{2\alpha}e^{-x^2}}{x-1}f(x)\mathrm{d}x - \lambda P\int_{-\infty}^{+\infty}\frac{x^{2\alpha}e^{-x^2}}{x-1}\chi_{[0,+\infty[}(x)f(x)\mathrm{d}x.\end{aligned} \tag{49}$$

The form $v$ is classical (semi-classical of class $s = 0$), it satisfies (24) and (27) with [10]

$$\begin{cases}\Phi(x) = x, \quad \Psi(x) = x-\alpha-1,\\ C_n(x) = -x+2n+\alpha, \quad D_n(x) = -1,\ n\ge 0.\end{cases} \tag{50}$$

The sequence $\{S_n\}_{n\ge 0}$ fulfils (5) with [6]

$$\xi_n = 2n+\alpha+1, \quad \rho_{n+1} = (n+1)(n+\alpha+1), \quad n\ge 0. \tag{51}$$

The regularity condition is $\alpha \ne -n,\ n\ge 1$.
First, we study the regularity of the form $u$.
From (7) and (2.11) in [6], we have for $n \ge 0$

$$S_n(1) = (-1)^n\sum_{k=0}^{n}\frac{(-1)^k\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(\alpha+k+1)}, \tag{52}$$

and

$$S_n^{(1)}(1) = (-1)^{n+1}\sum_{k=0}^{n+1}\frac{(-1)^k\Gamma(n+2)\Gamma(n+\alpha+2)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(\alpha+k+1)}b_{k-1}(\alpha), \tag{53}$$

where

$$b_n(\alpha) = \sum_{k=0}^{n}\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)}, \quad b_{-1}(\alpha) = 0.$$

By virtue of (8) and (52) – (53), we deduce

$$S_n(1,\lambda) = (-1)^n\Gamma(n+1)\Gamma(n+\alpha+1)c_n(\alpha,\lambda), \quad n \geq 0, \tag{54}$$

where

$$c_n(\alpha,\lambda) = \sum_{k=0}^{n} \frac{(-1)^k(1-\lambda b_{k-1})(\alpha)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(\alpha+k+1)}, \quad n \geq 0.$$

Then, $u$ is regular for every $\lambda \neq 0$ such that $c_n(\alpha,\lambda) \neq 0$, $n \geq 0$.

(18) and (54) give

$$a_n = (n+1)(n+\alpha+1)\frac{c_{n+1}(\alpha,\lambda)}{c_n(\alpha,\lambda)}, \quad n \geq 0. \tag{55}$$

Therefore, with (17), we obtain for $n \geq 0$

$$\begin{cases} \gamma_1 = -\lambda, \\ \gamma_{2n+2} = (n+1)(n+\alpha+1)\dfrac{c_{n+1}(\alpha,\lambda)}{c_n(\alpha,\lambda)}, \\ \gamma_{2n+3} = \dfrac{c_n(\alpha,\lambda)}{c_{n+1}(\alpha,\lambda)}. \end{cases} \tag{56}$$

Taking into account that the form $v$ is semi-classical and by virtue of Proposition 5, Proposition 8 and (50), the form $u$ is semi-classical of class $\tilde{s} = 2$ and fulfils (32) and (35) with

$$\begin{cases} \tilde{\Phi}(x) = x(x-1), \quad \tilde{\Psi}(x) = (x-1)(2x^2-2\alpha-1), \\ \tilde{C}_0(x) = -2x^3+2x^2+(2\alpha-1)x-2\alpha, \quad \tilde{D}_0(x) = 2(-x^2+\alpha+\lambda). \end{cases} \tag{57}$$

Now, we are going the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$.

$$\begin{cases} \tilde{C}_0(x) = -2x^3+2x^2+(2\alpha-1)x-2\alpha, \\ \tilde{C}_1(x) = 2(x-1)(-x^2+\alpha+2\lambda)+x, \\ \tilde{C}_{2n+2}(x) = 2(x-1)\left(-x^2+2n+\alpha+2-2(n+1)(n+\alpha+1)\dfrac{c_{n+1}(\alpha,\lambda)}{c_n(\alpha,\lambda)}\right)-x, \\ \tilde{C}_{2n+3}(x) = 2(x-1)\left(-x^2+2n+\alpha+2-2\dfrac{c_n(\alpha,\lambda)}{c_{n+1}(\alpha,\lambda)}\right)-x, \\ \tilde{D}_0(x) = 2(-x^2+\alpha+\lambda), \\ \tilde{D}_{2n+1}(x) = -2(x-1)^2, \\ \tilde{D}_{2n+2} = -2x^2+2\left(n+\alpha+1-(n+1)(n+\alpha+1)\dfrac{c_{n+1}(\alpha,\lambda)}{c_n(\alpha,\lambda)}-\dfrac{c_n(\alpha,\lambda)}{c_{n+1}(\alpha,\lambda)}\right). \end{cases}$$

(2) We study the problem (10), with $v := h_{\frac{1}{2}} o \tau_1 \mathcal{J}(\alpha,\beta)$ where $\mathcal{J}(\alpha,\beta)$ is the Jacobi form. This form has the following integral representation [10]

$$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^1 x^\alpha(1-x)^\beta f(x)\mathrm{d}x,\ \Re(\alpha),\ \Re(\beta) > -1,\ f \in \mathcal{P}. \tag{58}$$

Thus, using (23), we obtain the following integral representation of $u$

$$\begin{aligned}\langle u,f\rangle &= \lambda\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}\operatorname{sgn}x|x|^{2\alpha}(1-x^2)^{\beta-1}(x+1)f(x)\mathrm{d}x \\ &+ (1-\lambda\frac{\alpha+\beta+1}{\beta})f(1),\ \Re(\alpha)>-1,\ \Re(\beta)>0,\ f\in\mathcal{P}.\end{aligned} \tag{59}$$

The form $v$ is classical, it satisfies (24) and (27) with [10]

$$\begin{cases}\Phi(x)=x(x-1), \quad \Psi(x)=-(\alpha+\beta+2)x+\alpha+1, \\ C_n(x)=(2n+\alpha+\beta)x-n-\dfrac{(n+\alpha)(\alpha+\beta)}{2n+\alpha+\beta}, \\ D_n(x)=2n+\alpha+\beta+1,\ n\geq 0.\end{cases} \tag{60}$$

The sequence $\{S_n\}_{n\geq 0}$ fulfils (5) with [6]

$$\begin{cases}\xi_0=\frac{\alpha+1}{\alpha+\beta+2}, \quad \xi_{n+1}=\frac{1}{2}(\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}+1), n\geq 0, \\ \rho_{n+1}=\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}, n\geq 0.\end{cases} \tag{61}$$

The regularity conditions are $\alpha,\ \beta\neq -n,\ \alpha+\beta\neq -n,\ n\geq 1$.

Using (5) and (61), we get

$$S_n(1)=\frac{\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+2n+1)}, \quad n\geq 0. \tag{62}$$

From (6) and (61), we obtain by induction

$$S_n^{(1)}(1)=\frac{(\alpha+\beta+1)}{\Gamma(\alpha+\beta+2n+3)}d_n(\alpha,\beta), \quad n\geq 0, \tag{63}$$

where for $n\geq 0$

$$d_n(\alpha,\beta)=\begin{cases}\dfrac{1}{\alpha}\Big(\dfrac{\Gamma(\alpha+n+2)\Gamma(\alpha+\beta+n+2)}{\Gamma(\alpha+1)}-\dfrac{\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(\beta+n+2)}{\Gamma(\beta+1)}\Big), \alpha\neq 0, \\ \Gamma(n+1)\Gamma(n+\beta+2)\displaystyle\sum_{k=0}^{n}(\frac{1}{k+1}+\frac{1}{\beta+k+1}),\ \alpha=0.\end{cases}$$

By virtue of (8) and (62) − (63), we deduce

$$S_n(1,\lambda)=\frac{\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+2n+1)}e_n(\lambda,\alpha,\beta), \quad n\geq 0. \tag{64}$$

where for $n\geq 0$

$$e_n(\lambda,\alpha,\beta)=1-\lambda\frac{(\alpha+\beta+1)\Gamma(\beta+1)}{\Gamma(\beta+n+1)\Gamma(n+\alpha+\beta+1)}d_{n-1}(\alpha,\beta),\ d_{-1}(\alpha,\beta)=0.$$

Then, $u$ is regular for every $\lambda\neq 0$ such that

$$\lambda\neq\Big(\frac{(\alpha+\beta+1)\Gamma(\beta+1)}{\Gamma(\beta+n+1)\Gamma(n+\alpha+\beta+1)}d_{n-1}(\alpha,\beta)\Big)^{-1}, \quad n\geq 1. \tag{65}$$

(18) and (64) give

$$a_n = -\frac{(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\frac{e_{n+1}(\lambda,\alpha,\beta)}{e_n(\lambda,\alpha,\beta)}, \quad n \geq 0. \tag{66}$$

Then,with (17), we obtain for $n \geq 0$

$$\begin{cases} \gamma_1 = -\lambda, \\ \gamma_{2n+2} = -\dfrac{(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}\dfrac{e_{n+1}(\lambda,\alpha,\beta)}{e_n(\lambda,\alpha,\beta)}, \\ \gamma_{2n+3} = -\dfrac{(n+1)(\alpha+n+1)}{(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)}\dfrac{e_n(\lambda,\alpha,\beta)}{e_{n+1}(\lambda,\alpha,\beta)}. \end{cases} \tag{67}$$

Taking into account that the form $v$ is classical and by virtue of Proposition 5, the form $u$ is also semi-classical. It satisfies (32) and (35) with

$$\begin{cases} \tilde{\Phi}(x) = x(x-1)(x^2-1), \\ \tilde{\Psi}(x) = (x-1)\Big((2\alpha+2\beta-3)x^2+2\alpha+1\Big), \\ \tilde{C}_0(x) = -(x-1)\Big((2\alpha+2\beta+1)x^2+x+2\alpha\Big), \\ \tilde{D}_0(x) = 2(\alpha+\beta)x^2-2\alpha-2\lambda(\alpha+\beta+1). \end{cases} \tag{68}$$

From (60), we have

$$\begin{cases} \Phi(0) = 0, \quad \Phi(1) = 0, \\ X(1) = \beta-\lambda(\alpha+\beta+1), \quad X^{'}(1) = \alpha+\beta, \\ Y(1) = \beta-1. \end{cases}$$

Now it is enough to use Proposition 8 in order to obtain the following results:

(i) If $\lambda$ satisfies (65) and $\lambda \neq \frac{\beta}{\alpha+\beta+1}$, then the class of $u$ is $\tilde{s}=2$.

(ii) If $\lambda = \frac{\beta}{\alpha+\beta+1}$, then the class of $u$ is $\tilde{s}=1$ since $(X^{'}(1), Y(1)) \neq (0,0)$.

*Remarks.*

(i) The semi-classical orthogonal polynomials of class one satisfies (14) have been described in [5, 7].

(ii) If $\lambda = \frac{\beta}{\alpha+\beta+1}$, then from (59), we get for $\Re(\alpha) > -1$, $\Re(\beta) > 0$

$$\langle u, f\rangle = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\int_{-1}^{1} sgn\, x|x|^{2\alpha}(1-x^2)^{\beta-1}(x+1)f(x)\mathrm{d}x. \tag{69}$$

This result exist in [7, 12].

According to Proposition 9, (60) and (67), we have, for $n \geq 0$

$$\begin{cases} \tilde{C}_0(x) = -(x-1)\Big((2\alpha+2\beta+1)x^2 + x + 2\alpha\Big), \\ \tilde{C}_1(x) = (x-1)\Big((2\alpha+2\beta+1)x^2 - x - 2\alpha - 4\lambda(\alpha+\beta+1)\Big), \\ \tilde{C}_{2n+2}(x) = (x-1)\Big((4n+2\alpha+2\beta+3)x^2 + x - 2n - 2 \\ \qquad - 2\dfrac{(n+\alpha+1)(\alpha+\beta)}{2n+\alpha+\beta+2} - 4\dfrac{(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+2)}\dfrac{e_{n+1}(\lambda,\alpha,\beta)}{e_n(\lambda,\alpha,\beta)}\Big), \\ \tilde{C}_{2n+3}(x) = (x-1)\Big((4n+2\alpha+2\beta+5)x^2 - x - 2n - 2 \\ \qquad - 2\dfrac{(n+\alpha+1)(\alpha+\beta)}{2n+\alpha+\beta+2} - 4\dfrac{(n+1)(\alpha+n+1)}{(\alpha+\beta+2n+2)}\dfrac{e_n(\lambda,\alpha,\beta)}{e_{n+1}(\lambda,\alpha,\beta)}\Big), \\ \tilde{D}_0(x) = 2(\alpha+\beta)x^2 - 2\alpha - 2\lambda(\alpha+\beta+1), \\ \tilde{D}_{2n+1}(x) = 2(x-1)^2(\alpha+\beta+2n+1), \\ \tilde{D}_{2n+2} = 2\Big((2n+\alpha+\beta+2)x^2 - n - 1 - \dfrac{(n+\alpha+1)(\alpha+-\beta)}{2n+\alpha+\beta+2}\Big) \\ \qquad - 2\dfrac{(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+2)}\dfrac{e_{n+1}(\lambda,\alpha,\beta)}{e_n(\lambda,\alpha,\beta)} \\ \qquad - 2\dfrac{(n+1)(\alpha+n+1)}{(\alpha+\beta+2n+2)}\dfrac{e_n(\lambda,\alpha,\beta)}{e_{n+1}(\lambda,\alpha,\beta)}. \end{cases}$$

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